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The variety of convex 12-hedra revised

Yury L. Voytekhovsky[a,b]* **and Dmitry G. Stepenshchikov**[a]

[a]Laboratory for Mineralogy, Geological Institute of the Kola Science Centre, Russian Academy of Sciences, 14 Fersman Street, 184209 Apatity, Russia, and [b]Laboratory for Mathematical Investigations in Crystallography, Mineralogy and Petrography, High Technologies Centre, Kola Branch of Petrozavodsk State University, 3 Kosmonavtov Street, 184209 Apatity, Russia. Correspondence e-mail: voyt@geoksc.apatity.ru

The symmetry point group statistics for all combinatorially non-isomorphic convex 12-hedra (6384634 in total) are contributed in the paper. The numbers of 12-hedra with 11 to 19 vertices are revised. The most symmetrical shapes with 6 to 120 automorphism group orders (115 in total) are drawn in Schlegel diagrams and characterized by facet symbols and symmetry point groups.

1. Introduction

The numbers of combinatorial types of all convex 12-hedra were determined by Engel (1994). But his number 64445 of 12-hedra with 11 vertices does not correspond to the number 64439 of 11-hedra with 12 vertices. This is an obvious mistake because by duality these numbers should be equal. Besides, the number 268548 of 12-hedra with 12 vertices contradicts the number 268394 previously given by Duijvestijn & Federico (1981). Engel's numbers of 12-hedra with 13 to 19 vertices were never checked. Hence, our idea is to revise the combinatorial variety of convex 12-hedra.

2. Generation of polyhedra

We generated the polyhedra as their Schlegel diagrams. This is justified by the Steinitz (*every 3-connected planar graph can be realized as a polyhedron*) and Mani (*every combinatorial automorphism of a polyhedron is affinely realizable*) theorems (Steinitz, 1922; Steinitz & Rademacher, 1934). That is, there exists to each Schlegel diagram a 3-space realization of a polyhedron such that its edge graph is isomorphic to the Schlegel diagram while its symmetry point group is isomorphic to the automorphism group of the Schlegel diagram. As the simple 12-hedra (7595 in total) were already found, we used them to generate the non-simple 12-hedra by the ω-process of Fedorov's (1893) recurrence algorithm. It reduces any edge v_1–v_2 (joining vertices v_1 and v_2) if all facets containing v_1 but not v_2 have no common vertex with any facet containing v_2 but not v_1. Step by step, it allowed us to reduce the number of vertices from 20 (for simple 12-hedra) to 8. All the above operations and the determination of the symmetry point groups isomorphous to the related automorphism groups were done in accordance with our computer algorithms described in previous papers (Voytekhovsky, 2001*a*,*b*; Voytekhovsky & Stepenshchikov, 2002*a*,*b*, 2003*a*,*b*). Briefly, we calculate the automorphism group order of a polyhedron as the number of different vertex reindexings that save its adjacency matrix. Afterwards, any such reindexing is identified as a symmetry element (plane, axis, inversion axis or center) in accordance with some rules.

3. Results and discussion

The numbers of combinatorially different types, automorphism group order (a.g.o.) and symmetry point group (s.p.g.) statistics of 12-hedra with various numbers of vertices are given in Table 1. As can be seen, the numbers of polyhedra with 8 to 12 and 20 vertices agree with those by Duijvestijn & Federico (1981). But the numbers of polyhedra with 11 to 19 vertices are less than the numbers given by Engel (1994). As in the cases of 4- to 11-hedra and simple 13- to 15-hedra, the shapes of 1, m, 2 and $mm2$ symmetry point groups prevail among the 12-hedra, and the trivial shapes include the overwhelming majority.

The most symmetrical 12-hedra with automorphism group orders not less than 6 are drawn in Schlegel diagrams in Fig. 1. A projection is usually made along the main symmetry axis onto the orthogonal facet, if any. If this cannot be done, we draw the Schlegel diagrams into one of the n-gonal facets with highest n preserving as much as possible of the symmetry.

As in our previous papers, we use the facet symbols [in brackets] to lexicographically order the polyhedra. They give the numbers of 3-, 4, . . . , n-gonal facets in sequence.

8 vertices: [12] $\bar{4}2m$: 1, $\bar{3}m$: 2, $\bar{4}3m$: 3, $6/mmm$: 4. **10 vertices:** [84] $\bar{4}2m$: 5, $4mm$: 6, 7, $4/mmm$: 8; [10,02] $\bar{5}m$: 9. **11 vertices:** [66] 32: 10, $3m$: 11–15, $\bar{6}m2$: 16–18; [903] $3m$: 19–22. **12 vertices:** [48] mmm: 23, 24, $\bar{4}2m$: 25–27; [804] $\bar{4}2m$: 28; [11,00000001] $11m$: 29. **14 vertices:** [0,12] $\bar{3}m$: 30, $\bar{6}m2$: 31, $\overline{12}m2$: 32, $m\bar{3}m$: 33; [363] $3m$: 34–39; [444] mmm: 40, $\bar{4}2m$: 41, $4mm$: 42, 43; [606] $3m$: 44, $\bar{3}m$: 45–47, $\bar{6}m2$: 48, 49; [6303] $3m$: 50–54; [8004] $\bar{4}2m$: 55. **15 vertices:** [0,10,2] $\overline{10}m2$: 56; [507] $5m$: 57; [55100001] $5m$: 58. **16 vertices:** [084] mmm: 59, 60, $\bar{4}2m$: 61–63; [4404] mmm: 64, $\bar{4}2m$: 65, 66; [80004] $\bar{4}2m$: 67. **17 vertices:** [066] 32: 68, $3m$: 69, $\bar{6}m2$: 70; [0903] $3m$: 71, 72; [309] $3m$: 73, $\bar{6}$: 74; [3333] $3m$: 75–79; [36003] $3m$: 80; [6006] 32: 81, $3m$: 82, 83; [60303] $3m$: 84, 85. **18 vertices:** [048] mmm: 86, $4/mmm$: 87; [0804] $4/mmm$: 88; [4044] $\bar{4}2m$: 89, $4mm$: 90; [424002] mmm: 91; [44004] $\bar{4}2m$: 92. **20 vertices:** [0,0,12] $\bar{3}\bar{5}m$: 93; [0363] $\bar{6}m2$: 94; [0444] mmm: 95, $\bar{4}2m$: 96–98; [0606] 32: 99, mmm: 100, $\bar{3}m$: 101; [06303] $3m$: 102; [064002] mmm: 103; [08004] $\bar{4}2m$: 104; [0,10,000002] $10/mmm$: 105; [3036] $\bar{6}$: 106; [30603] $3m$: 107; [33033] $3m$: 108; [333003] $3m$: 109, 110; [4008] $\bar{4}2m$: 111; [420402] mmm: 112; [440004] $\bar{4}2m$: 113; [50150001] $5m$: 114; [60006] $\bar{3}m$: 115.

4. Conclusions

Up to now, all the varieties of 4- to 12- and simple 13- to 15-hedra have been enumerated and characterized by facet symbols and symmetry point groups. The most symmetrical shapes are drawn as Schlegel diagrams. The next steps are to generate and characterize all non-simple 13- and simple 16-hedra in the same way.

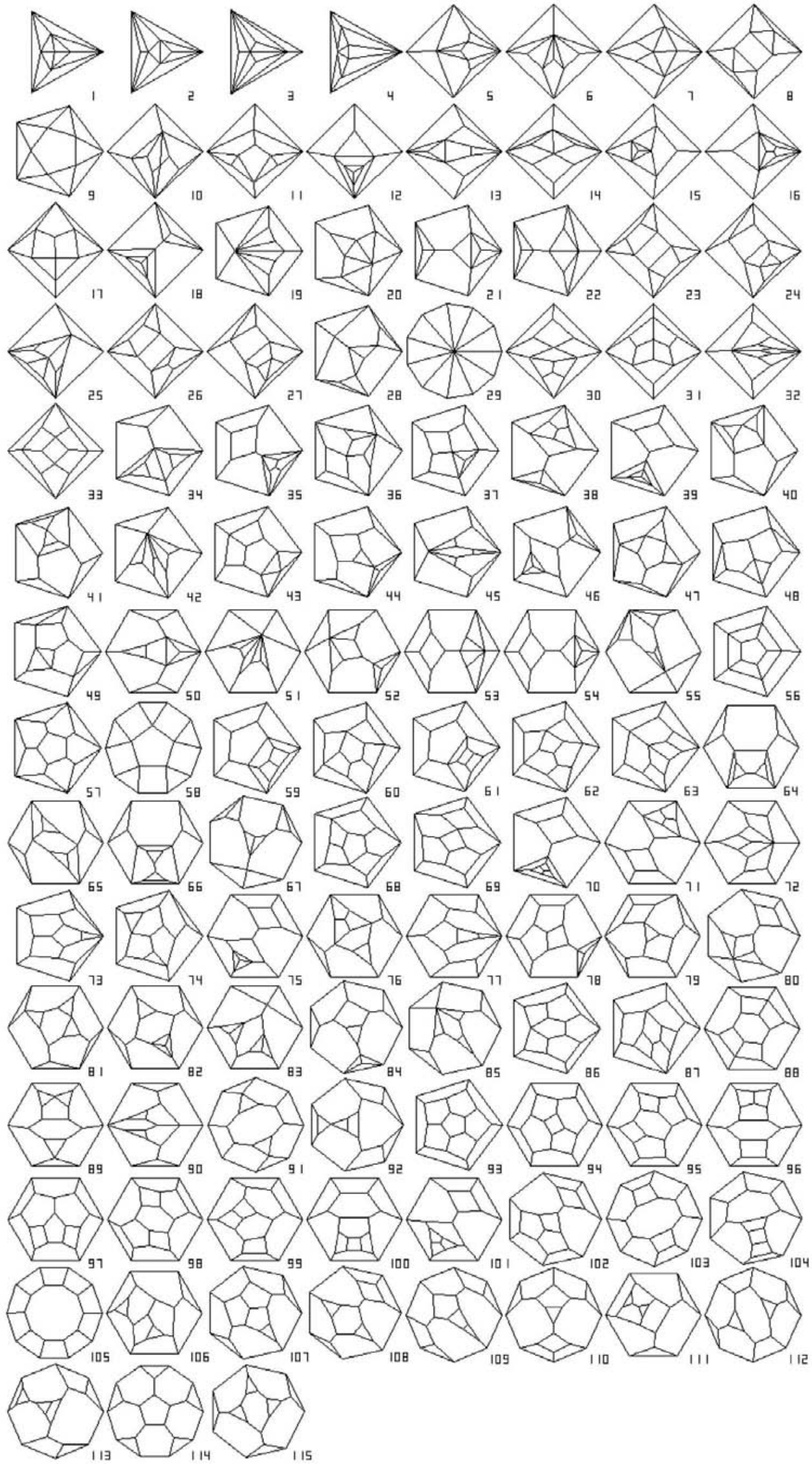

Figure 1
The most symmetrical 12-hedra with a.g.o. not less than 6.

Table 1
The automorphism group order and symmetry point group statistics for combinatorially non-isomorphic convex 12-hedra.

		Vertices													
a.g.o.	s.p.g.	8	9	10	11	12	13	14	15	16	17	18	19	20	Total
1	1	2	449	8331	63080	265253	704267	1255238	1548735	1329899	783543	301763	68697	6756	6336013
2	2	1	25	109	241	635	695	1458	899	1577	545	846	113	146	7290
	m	4	74	347	1076	2393	4282	6109	7242	7129	5580	3708	1598	597	40139
	$\bar{1}$			3		24		64		68		38		4	201
3	3				4			14			10			1	29
4	*mm*2	3	10	17	25	49	58	86	73	112	53	84	46	53	669
	2/*m*			6		18		23		35		18		10	110
	222			3		10		10		16		5		3	47
	4							1							1
	$\bar{4}$			1		5		3		8		1		2	20
6	3*m*				9			12			14			5	40
	32				1						2			1	4
	$\bar{6}$										1			1	2
8	*mmm*					2		1		3		2		4	12
	4*mm*			2				2				1			5
	$\bar{4}2m$	1		1		4		2		6		2		6	22
10	5*m*								2					1	3
12	$\bar{3}m$	1						4						2	7
	$\bar{6}m2$				3			3			1			1	8
16	4/*mmm*			1								2			3
20	$\overline{10}m2$								1						1
	$\bar{5}m$			1											1
22	11*m*					1									1
24	6/*mmm*	1													1
	$\bar{4}3m$	1													1
	$\overline{12}m2$							1							1
40	10/*mmm*													1	1
48	$m\bar{3}m$							1							1
120	$\bar{3}\bar{5}m$													1	1
Total		14	558	8822	64439	268394	709302	1263032	1556952	1338853	789749	306470	70454	7595	6384634

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